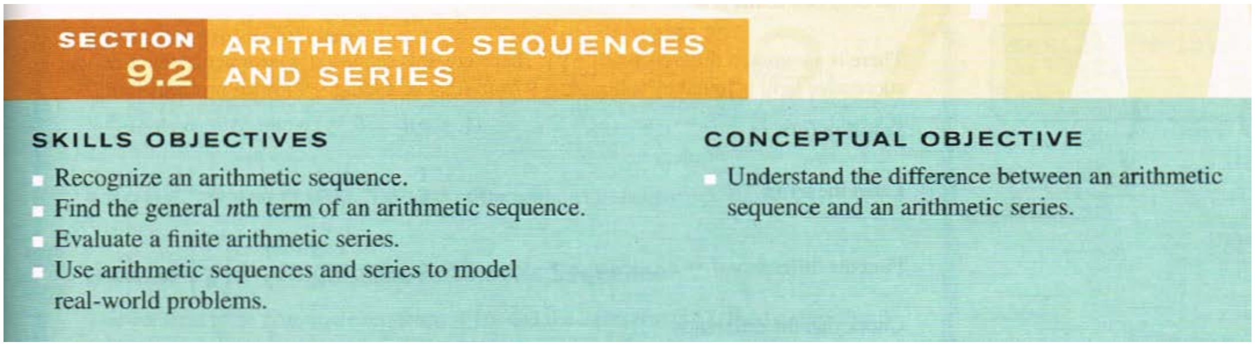
Ch 9.2 Arithmetic Sequences and Series


SECTION 9.2 ARITHMETIC SEQUENCES AND SERIES

SKILLS OBJECTIVES
- Recognize an arithmetic sequence.
- Find the general *n*th term of an arithmetic sequence.
- Evaluate a finite arithmetic series.
- Use arithmetic sequences and series to model real-world problems.

CONCEPTUAL OBJECTIVE
- Understand the difference between an arithmetic sequence and an arithmetic series.

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SKILLS OBJECTIVES

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CONCEPTUAL OBJECTIVE

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Arithmetic Sequences

The word arithmetic (with emphasis on the third syllable) often implies adding or subtracting of numbers. **Arithmetic sequences are sequences whose terms are found by adding a constant to each previous term.** The sequence 1, 3, 5, 7, 9, . . . is arithmetic because each successive term is found by adding 2 to the previous term.

DEFINITION Arithmetic Sequences

A sequence is arithmetic if each term in the sequence is found by adding a real number d to the previous term, so that $a_{n+1} = a_n + d$.

Because $a_{n+1} - a_n = d$ **, the number d is called the common difference.**

EXAMPLE 1

Identifying the Common Difference in Arithmetic Sequences

Determine whether each sequence is arithmetic. If so, find the common difference for each of the arithmetic sequences.

a. 5, 9, 13, 17, ...

Solution (a):

Label the terms. $a_1 = 5, a_2 = 9, a_3 = 13, a_4 = 17, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = 9 - 5 = \boxed{4}$

Check that the difference for the next two terms is also 4.

$d = a_3 - a_2 = 13 - 9 = 4$

$d = a_4 - a_3 = 17 - 13 = 4$

There is a common difference of $\boxed{4}$. Therefore, this sequence is arithmetic and each successive term is found by adding 4 to the previous term.

Determine whether each sequence is arithmetic. If so, find the common difference for each of the arithmetic sequences.

b. 18, 9, 0, -9, ...

Solution (b):

Label the terms. $a_1 = 18, a_2 = 9, a_3 = 0, a_4 = -9, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = 9 - 18 = \boxed{-9}$

Check that the difference for the next two terms is also -9.

$d = a_3 - a_2 = 0 - 9 = -9$

$d = a_4 - a_3 = -9 - 0 = -9$

There is a common difference of $\boxed{-9}$. Therefore, this sequence is arithmetic and each successive term is found by subtracting 9 from (that is, adding -9 to) the previous term.

Determine whether each sequence is arithmetic. If so, find the common difference for each of the arithmetic sequences.

c. $\frac{1}{2}, \frac{5}{4}, 2, \frac{11}{4}, \ldots$

Solution (c):

Label the terms. $a_1 = \frac{1}{2}, a_2 = \frac{5}{4}, a_3 = 2, a_4 = \frac{11}{4}, \ldots$

Find the difference $d = a_{n+1} - a_n$. $d = a_2 - a_1 = \frac{5}{4} - \frac{1}{2} = \boxed{\frac{3}{4}}$

Check that the difference for the next two terms is also $\frac{3}{4}$.

$$d = a_3 - a_2 = 2 - \frac{5}{4} = \frac{3}{4}$$

$$d = a_4 - a_3 = \frac{11}{4} - 2 = \frac{3}{4}$$

There is a common difference of $\boxed{\frac{3}{4}}$. Therefore, this sequence is arithmetic and each successive term is found by adding $\frac{3}{4}$ to the previous term.

• *YOUR TURN*

Find the common difference for each of the arithmetic sequences.

a. 7, 2, -3, -8, ...

b. $1, \frac{5}{3}, \frac{7}{3}, 3, \ldots$

■ **Answer: a.** -5 **b.** $\frac{2}{3}$

The General (nth) Term of an Arithmetic Sequence

To find a formula for the general, or , nth, term of an arithmetic sequence, write out the first several terms and look for a pattern.

First term, $n = 1$.	a_1
Second term, $n = 2$.	$a_2 = a_1 + d$
Third term, $n = 3$.	$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$
Fourth term, $n = 4$.	$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$

In general, the nth term is given by $a_n = a_1 + (n - 1)d$.

THE nTH TERM OF AN ARITHMETIC SEQUENCE

The nth term of an arithmetic sequence with common difference d is given by
aA = a,+(n—1)d forn1

$$a_n = a_1 + (n - 1)d \qquad \text{for } n \geq 1$$

EXAMPLE 2 Finding the nth Term of an Arithmetic Sequence

Find the 13th term of the arithmetic sequence 2, 5, 8, 11

Solution:

Identify the common difference.	$d = 5 - 2 = 3$
Identify the first ($n = 1$) term.	$a_1 = 2$
Substitute $a_1 = 2$ and $d = 3$ into $a_n = a_1 + (n - 1)d$.	$a_n = 2 + 3(n - 1)$
Substitute $n = 13$ into $a_n = 2 + 3(n - 1)$.	$a_{13} = 2 + 3(13 - 1) = \boxed{38}$

Technology Tip

[2nd] [LIST] [▶] [OPS] [▼] [5:seq(]
[ENTER] 2 [+] 3 [(] [ALPHA] [N]
[–] [1] [)] [,] [ALPHA] [N] [,] 13 [,]
13 [,] 1 [)] [ENTER]

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seq(2+3(N-1),N,1
3,13,1)
                {38}
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• **YOUR TURN** **Find the 10th term of the arithmetic sequence 3. 10, 17, 24**

■ **Answer:** 66

EXAMPLE 3 Finding the Arithmetic Sequence

The 4th term of an arithmetic sequence is 16, and the 21st term is 67. Find $a_1,\ and\ d$ and construct the sequence.

Solution:

Write the 4th and 21st terms.	$a_4 = 16$ and $a_{21} = 67$
Adding d 17 times to a_4 results in a_{21}.	$a_{21} = a_4 + 17d$
Substitute $a_4 = 16$ and $a_{21} = 67$.	$67 = 16 + 17d$
Solve for d.	$\boxed{d = 3}$
Substitute $d = 3$ into $a_n = a_1 + (n - 1)d$.	$a_n = a_1 + 3(n - 1)$
Let $a_4 = 16$.	$16 = a_1 + 3(4 - 1)$
Solve for a_1.	$\boxed{a_1 = 7}$

The arithmetic sequence that starts at 7 and has a common difference of 3 is $\boxed{7, 10, 13, 16, \ldots}$.

• YOUR TURN

Construct the arithmetic sequence whose 7th term is 26 and whose 13th term is 50.

■ **Answer:** 2, 6, 10, 14, . . .

The Sum of an Arithmetic Sequence

What is the sum of the first 100 counting numbers

$$1 + 2 + 3 + 4 + \cdots + 99 + 100 = ?$$

If we write this sum twice (one in ascending order and one in descending order) and add, we get 100 pairs of 101.

$$\begin{array}{r} 1 + \ \ 2 + \ \ 3 + \ \ 4 + \cdots + \ 99 + 100 \\ \underline{100 + \ 99 + \ 98 + \ 97 + \cdots + \ \ 2 + \ \ \ 1} \\ 101 + 101 + 101 + 101 + \cdots + 101 + 101 = 100\,(101) \end{array}$$

Since we added twice the sum, we divide by 2.

$$1 + 2 + 3 + 4 + \cdots + 99 + 100 = \frac{(101)(100)}{2} = 5{,}050$$

Now, let us develop the sum of a general arithmetic series.

The sum of the first n terms of an arithmetic sequence is called the nth partial sum, or finite arithmetic series, and is denoted by S_n .

An arithmetic sequence can be found by starting at the first term and adding the common difference to each successive term, and so the **nth partial sum**, or **finite series**, can be found the same way, but terminating the sum at the nth term:

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots$$
$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_n)$$

Similarly, we can start with the nth term and find terms going backward by subtracting the common difference until we arrive at the first term:

$$S_n = a_n + a_{n-1} + a_{n-2} + a_{n-3} + \cdots$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + (a_1)$$

Add these two representations of the nth partial sum. Notice that the d terms are eliminated:

$$S_n = a_1 + (a_1 + d) + (a_1 + 2d) + (a_1 + 3d) + \cdots + (a_n)$$
$$S_n = a_n + (a_n - d) + (a_n - 2d) + (a_n - 3d) + \cdots + (a_1)$$
$$2S_n = \underbrace{(a_1 + a_n) + (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n)}_{n(a_1 + a_n)}$$

$$2S_n = n(a_1 + a_n) \qquad \text{or} \qquad S_n = \frac{n}{2}(a_1 + a_n)$$

Let $a_n = a_1 + (n - 1)d$.

$$S_n = \frac{n}{2}[a_1 + a_1 + (n - 1)d]$$

$$= \frac{n}{2}[2a_1 + (n - 1)d] = na_1 + \frac{n(n - 1)d}{2}$$

DEFINITION Evaluating a Finite Arithmetic Series

The sum of the first n terms of an arithmetic sequence, called a finite arithmetic series, is given by the formula

$$S_n = \frac{n}{2}(a_1 + a_n) \quad n \geq 2$$

Study Tip

S_n can also be written as

$$S_n = \frac{n}{2}[2a_1 + (n - 1)d].$$

EXAMPLE 4 Evaluating a Finite Arithmetic Series

Evaluate the finite arithmetic series $\sum_{k=1}^{100} k$.

Solution:

Expand the arithmetic series. $$\sum_{k=1}^{100} k = 1 + 2 + 3 + \cdots + 99 + 100$$

This is the sum of an arithmetic sequence of numbers with a common difference of 1.

Identify the parameters of the arithmetic sequence. $a_1 = 1$, $a_n = 100$, and $n = 100$

Substitute these values into $S_n = \frac{n}{2}(a_1 + a_n)$. $$S_{100} = \frac{100}{2}(1 + 100)$$

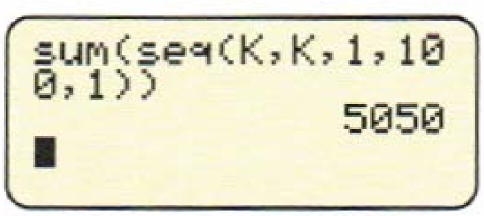
Simplify. $\boxed{S_{100} = 5050}$

The sum of the first 100 natural numbers is 5050.

Technology Tip

To find the sum of the series $\sum_{k=1}^{100} k$, press

[2nd] [LIST] [▶] [MATH] [▼]
[5:sum(] [ENTER] [2nd] [LIST]
[▶] [OPS] [▼] [5:seq(] [ENTER]
[ALPHA] [K] [,] [ALPHA] [K] [,]
1 [,] 100 [,] 1 [)] [)] [ENTER]



• **YOUR TURN** **Evaluate the following finite arithmetic series.**

a. $\sum_{k=1}^{30} k$ **b.** $\sum_{k=1}^{20} (2k + 1)$

Answer: a. 465 **b.** 440

EXAMPLE 5 Finding the nth Partial Sum of an Arithmetic Sequence

Find the sum of the first 20 terms of the arithmetic sequence 3, 8, 13, 18, 23

Solution:

Recall the partial sum formula. $S_n = \frac{n}{2}(a_1 + a_n)$

Find the 20th partial sum of this arithmetic sequence. $S_{20} = \frac{20}{2}(a_1 + a_{20})$

Recall that the general nth term of an arithmetic sequence is given by: $a_n = a_1 + (n - 1)d$

Note that the first term of the arithmetic sequence is 3. $a_1 = 3$

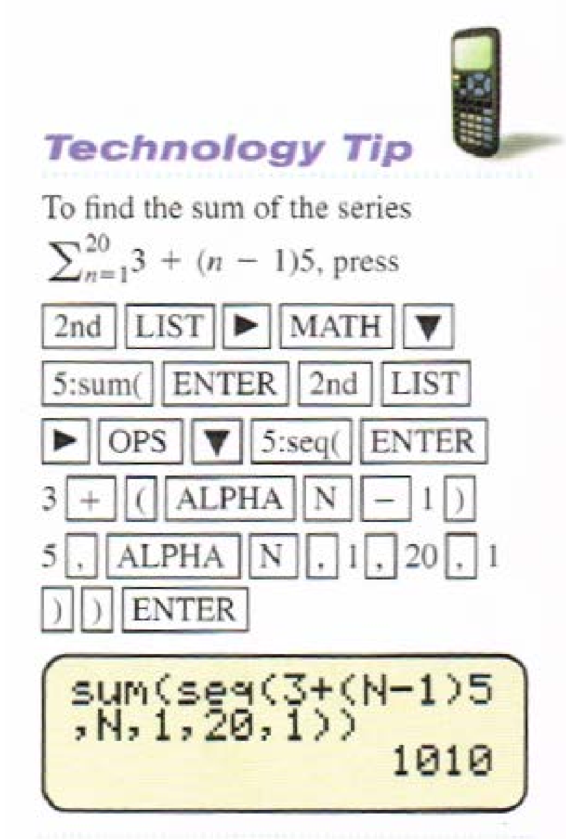
This is an arithmetic sequence with a common difference of 5. $d = 5$

Substitute $a_1 = 3$ and $d = 5$ into $a_n = a_1 + (n - 1)d$. $a_n = 3 + (n - 1)5$

Substitute $n = 20$ to find the 20th term. $a_{20} = 3 + (20 - 1)5 = 98$

Substitute $a_1 = 3$ and $a_{20} = 98$ into the partial sum. $\boxed{S_{20} = 10(3 + 98) = 1010}$

The sum of the first 20 terms of this arithmetic sequence is 1010.



Technology Tip

To find the sum of the series $\sum_{n=1}^{20} 3 + (n - 1)5$, press

[2nd] [LIST] [▶] [MATH] [▼]
[5:sum(] [ENTER] [2nd] [LIST]
[▶] [OPS] [▼] [5:seq(] [ENTER]
3 [+] [(] [ALPHA] [N] [−] 1 [)]
5 [,] [ALPHA] [N] [,] 1 [,] 20 [,] 1
[)] [)] [ENTER]

• YOUR TURN **Find the sum of the first 25 terms of the arithmetic sequence 2, 6, 10, 14, 18**

•Answer: 1250

Applications

EXAMPLE 6 Marching Band Formation

Suppose a band has 18 members in the first row. 22 members in the second row, and 26 members in the third row and continues with that pattern for a total of nine rows. How many marchers are there all together?

David Young-Wolff/PhotoEdit

UC Berkeley marching band

Solution:

The number of members in each row forms an arithmetic sequence with a common difference of 4, and the first row has 18 members. $a_1 = 18 \quad d = 4$

Calculate the nth term of the sequence $a_n = a_1 + (n - 1)d$. $a_n = 18 + (n - 1)4$

Find the 9th term $n = 9$. $a_9 = 18 + (9 - 1)4 = 50$

Calculate the sum $S_n = \frac{n}{2}(a_1 + a_n)$ of the nine rows.

$$S_9 = \frac{9}{2}(a_1 + a_9)$$
$$= \frac{9}{2}(18 + 50)$$
$$= \frac{9}{2}(68)$$
$$= \boxed{306}$$

There are 306 members in the marching band.

• YOUR TURN

Suppose a bed of tulips is arranged in a garden so that there are 20 tulips in the first row. 26 tulips in the second row, and 32 tulips in the third row and the rows continue with that pattern for a total of 8 rows. How many tulips are there all together?

•Answer: 328

SECTION 9.2 SUMMARY

fin this section, arithmetic sequences were defined as sequences of which each successive term is found by adding the same constant d to the previous term. Formulas were developed for the general, or nth. term of an arithmetic sequence, and for the nth partial sum of an arithmetic sequence, also called a finite arithmetic series.

$$a_n = a_1 + (n-1)d \qquad n \geq 1$$

$$S_n = \frac{n}{2}(a_1 + a_n) = na_1 + \frac{n(n-1)}{2}d$$

SECTION 9.2 EXERCISES

▪ SKILLS

in Exercises 1—10, determine whether the sequence is arithmetic. If it is, find the common difference.

1. $2, 5, 8, 11, 14, \ldots$

2. $9, 6, 3, 0, -3, -6, \ldots$

3. $1^2 + 2^2 + 3^2 + \cdots$

4. $1! + 2! + 3! + \cdots$

5. $3.33, 3.30, 3.27, 3.24, \ldots$

6. $0.7, 1.2, 1.7, 2.2, \ldots$

7. $4, \frac{14}{3}, \frac{16}{3}, 6, \ldots$

8. $2, \frac{7}{3}, \frac{8}{3}, 3, \ldots$

9. $10^1, 10^2, 10^3, 10^4, \ldots$

10. $120, 60, 30, 15, \ldots$